Universality and non-universality in behavior of self-repairing random networks

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We numerically study one-parameter family of random single-cluster systems. A finite-concentration topological phase transition from the net-like to the tree-like phase (the latter is without a backbone) is present in all models of the class. Correlation radius index ν_B of the backbone in the net-like phase; graph dimensions – $d_{\min}$ of the tree-like phase, and $D_{\min}$ of the backbone in the net-like phase appear to be universal within the accuracy of our calculations, while the backbone fractal dimension D_B is not universal: it depends on the parameter of a model.

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Properties of disordered networks, such as porous materials, are of great interest from both practical and theoretical point of view (see [1]). The standard percolation models (see [2, 3]) can not adequately describe many of these systems, in particular the "single-cluster" ones, which are obligatory connected (e.g., because of mechanical stability requirements) and do not allow for detachment of finite clusters. In our earlier paper [4] we have introduced and studied a simple model of self-repairing bond-percolation (SRBP) in which all finite clusters arising in the process of gradual destruction of randomly chosen bonds in the system are immediately "repaired" by means of regeneration of that very bond whose destruction has lead to violation of connectivity. The system therefore remains connected and constitutes a single large cluster at all accessible concentrations. The SRBP-model turned out to be analytically tractable by means of a partial mapping onto the standard bond-percolation model; we were able to prove the existence of a finite-density topological phase transition, at which the backbone of the system vanishes and the system occurs in a fragile tree-like state with zero mechanical rigidity and zero electric conductivity. The backbone density P_B and correlation radius ξ_B slightly above the transition follow the scaling laws

$$P_B(p) \propto (p - p_c)^{\beta_B}, \qquad \xi_B(p) \propto (p - p_c)^{-\nu_B}. \quad (1)$$

The critical concentration of bonds p_c and relevant critical exponents could be related to those of standard bond percolation (though not being necessarily equal to them). In particular, in two-dimensional lattices

$$\beta_B^{(\mathrm{SRBP})} \equiv \beta_B^{(\mathrm{perc})} = 0.4757(10), \quad (2)$$

$$\nu_B^{(\mathrm{SRBP})} \equiv \nu^{(\mathrm{perc})} = 4/3. \quad (3)$$

The numerical value for $\beta_B^{(\mathrm{perc})}$ in $2D$, given in(2), is taken from [7], while the exact value of $\nu^{(\mathrm{perc})}$ in $2D$ was first obtained in [8]. The fractal dimension of the backbone is generally related to β_B and ν_B

$$D_B = D - \beta_B/\nu_B. \quad (4)$$

For the SRBP model it is also the same, as in percolation. In $2D$:

$$D_B^{(\mathrm{SRBP})} \equiv D_B^{(\mathrm{perc})} = 1.6432(8). \quad (5)$$

From the topological point of view the difference between the tree-like phase and the usual net-like one is the abundant presence of arbitrary large (up to the system size) cycles in the graph of the net-like system and exponential decay of concentration of large cycles on the graph of the tree-like system.

One can easily envisage many other models of single-cluster random networks. We were not able, however, to develop an analytical approach to any physically reasonable model except SRBP. The question about universality of the discovered phase transition (and, moreover, the very existence of the transition in a particular model) is very important and nontrivial. In contrast to standard percolation models, the connectivity constraint (common for all single-cluster models) is essentially nonlocal. This increased complexity of the model makes the existence of relations of single-cluster models to local field theories (similar to the well-known relation between the standard percolation and the Potts-model, see [5] and [3]) highly improbable; we cannot therefore proclaim the universality basing on standard renormalization-group arguments. Note, that the invasion percolation (see [3, 6]), which also involves a nonlocal constraint in its definition, does not belong to the standard percolation universality class and is characterized by its own critical indices. Thus any new single-cluster model introduces a new puzzle of its own, and it is very interesting and important to explore different patterns of their behavior and try to establish some order in the corresponding zoology. In this paper we start this program, and study the non-universality of critical exponents.

Our first step was numerical exploration of the self-repairing site-percolation (SRSP) on a square lattice, where randomly chosen *sites* of the lattice (not *bonds*, as in SRBP) are gradually removed and regenerated each time when the removal causes disconnection of a finite cluster from the mainland. Numerical simulation reveals

a well-defined phase transition at the site-concentration $x_c = 0.608(1)$ with the critical index of the backbone density $\beta_B^{(\text{SRSP})} = 0.463(2)$, which is distinct from the result (2) for the SRBP-model. The latter difference is a definite manifestation of the non-universality of the phase transition. On the other hand, our simulation did not show any reliable difference between the SRSP index of correlation radius: $\nu_B^{(\text{SRSP})} = 1.337(8)$ and $\nu^{(\text{SRBP})} = 4/3$. For the fractal dimension, using (4), we obtain

$$D_B^{(\text{SRSP})} = 1.653(2), \tag{6}$$

also different from $D_B^{(\text{SRBP})}$.

Another important fractal characteristic of the critical backbone on the spatial scale $R \ll \xi$ is the graph dimension $D_{\min}$, describing the dependence of average "chemical distance" $\ell(R)$ (that is, the length of the shortest path between two sites of a backbone, separated by euclidean distance R)

$$\ell_B(R) \propto R^{D_{\min}}. \tag{7}$$

As we have shown in [4],

$$D_{\min}^{(\text{SRBP})} \equiv D_{\min}^{(\text{perc})} = 1.13(2), \tag{8}$$

where the numerical value (for $2D$) was taken from [9]. Note that the graph dimension of the percolation backbone coincides with that of the entire infinite cluster (see [3]). Our simulations for the SRSP-model give $D_{\min}^{(\text{SRSP})} = 1.136(10)$, that, again, is not different from the SRBP value (8) within our accuracy.

In contrast with the SRBP-model, where the tree-like phase can only exist in the range $p_{\text{tree}} < p < p_c$, bounded from below by $p_{\text{tree}} > 0$ (see [4]), in SRSP-model the quasi-tree cluster can have arbitrary low concentration ($0 < x < x_c$). For $x \to 0$ the concentration of cycles vanish very rapidly, so that the patterns looks like a tree already slightly below x_c. In Fig.1 reasonably large randomly chosen fragments of a quasi-tree samples with $x = 0.5$ and $x = 0.25$ are shown. It demonstrates that the cycles die out very fast with lowering x: the sample with $x = 0.25$ is already practically a tree.

The tree-like phase is fractal in the entire range $x < x_c$, so that the chemical distance $\ell_T(R)$ in this phase obeys

$$\ell_T(R) \propto R^{d_{\min}}, \qquad 1 < d_{\min} < 2. \tag{9}$$

As was shown (both numerically and analytically) in [4], for SRBP-model $d_{\min}$ does not depend on p throughout the tree-like phase and exactly coincides with the similar index $d_{\min}^{(\text{MST})}$ of the Minimal Spanning Trees ensemble (MST) on a lattice:

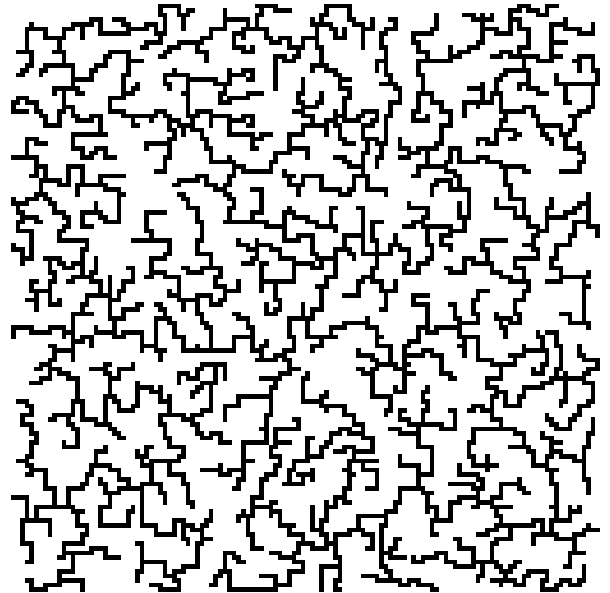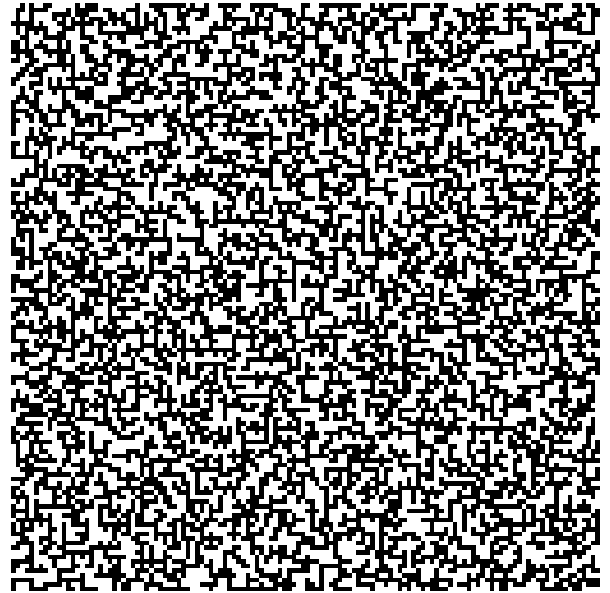$$d_{\min}^{(\text{SRBP})} \equiv d_{\min}^{(\text{MST})} = 1.22(1). \tag{10}$$

FIG. 1: SRSP-patterns in the tree-like phase: upper panel, $x = 0.5$, here few small cycles are still visible; lower panel, $x = 0.25$: concentration of cycles is already so low, that we do not see any of them in this fragment of a quasi-tree.

The latter numerical value of $d_{\min}^{(\text{MST})}$ in $2D$ was obtained in [10, 11] (see, also [4]).

To determine $d_{\min}$ and $D_{\min}$ for SRSP model in this paper we simulated systems of size 1024×1024 with open boundary conditions. Statistics on average Euclidean displacement $\langle R \rangle$ as a function of chemical distance ℓ was gathered by performing breadth-first searches on points from the central part of the system. About 10^7 starting points were processed, and plain power fits $\langle R \rangle \sim \ell^{1/d_{\min}}$
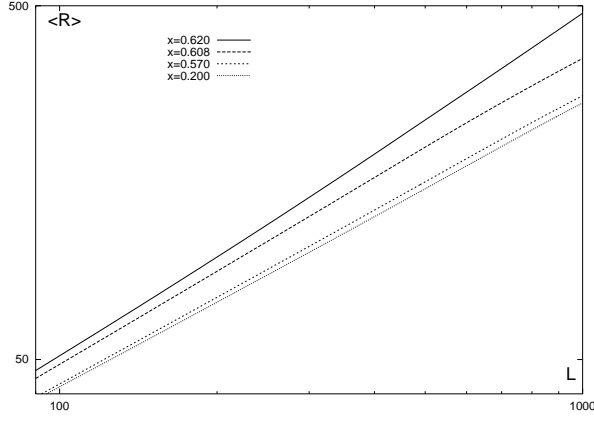
FIG. 2: Log-log plots for root mean square euclidean displacement R vs. chemical distance ℓ for the SRSP-model on square lattice with four different concentrations: $x = 0.62$ (above the threshold); $x = 0.608$ (at the threshold); $x = 0.57$ and $x = 0.20$ (both below the threshold). Within the accuracy of our calculations, the slope of the first curve corresponds to $d_{\min} = 1.0002$ (practically, unity), the second curve – to $D_{\min} = 1.136(10)$. The third and the fourth curves both have the same slope, corresponding to $d_{\min} = 1.226(12)$ (practically, the same as $d_{\min}^{(MST)}$).

were applied in the stable region $500 < \ell < 900$ not significantly affected by the boundary. Our simulations give, again, x-independent

$$d_{\min}^{(SRSP)} = 1.226(12), \qquad (11)$$

throughout the tree-like range of x. This value, within our accuracy, does not differ from $d_{\min}^{(MST)}$. To our knowledge, however, there is no obvious relation between quasitrees, appearing in the tree-like phase of the SRSP, and the MST ensemble.

At very low concentration $x \ll 1$ the tree-like phase of the SR(S/B)P-models apparently acquire a new spatial scale $\xi_C(x) \sim x^{-\nu_C} \gg 1$, – the density correlation length, that diverges as $x \to 0$. On the scales $1 \ll L \ll \xi_C$ the quasi-tree, besides being a "chemical fractal" (with nontrivial $d_{\min} > 1$) becomes also a "density fractal" with nontrivial fractal dimension D_C (see Fig.3). Properties of this low-density phase and corresponding critical indices will be studied in a separate publication.

For further study of the non-universality we have considered a one-parameter family of hybridized SR(S/B)P-models, in which, at each step of the process, with probability $1-Q$ a randomly chosen bond is removed (and restored, if necessary) and, with probability Q, a randomly chosen site together with all adjacent bonds is removed (and also restored, if necessary). In Fig.4 the concentration dependence of the backbone density $P_B(p)$ is shown for five different values of the parameter Q. For three of them ($Q = 0$, $Q = 0.5$, and $Q = 1$) the critical exponents β_B, ν and ω were accurately extracted from the data.
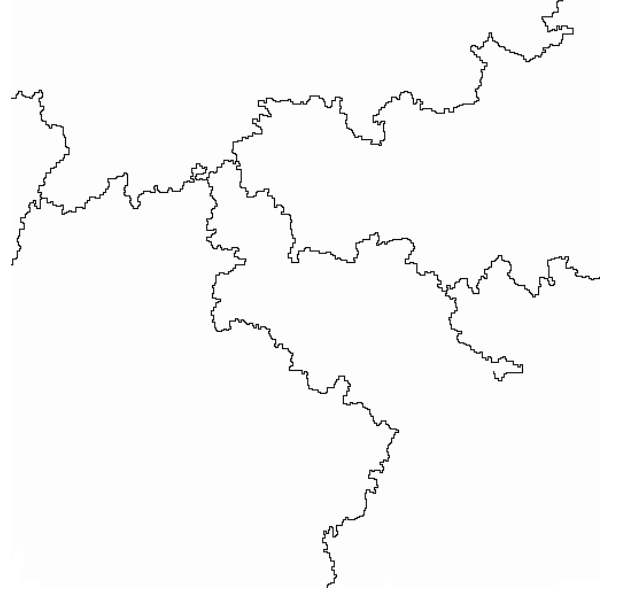
We have devised an efficient algorithm for simulations

FIG. 3: SRSP-pattern at low density $x = 0.0125$: not only a tree, but apparently also a density fractal.

Q	p_c	β_B	D_B	ω
0	$(3/4)(\sqrt{3}-1)$	0.4757(10)	1.6432(8)	1.85(1)
0.5	0.420476(1)	0.468(2)	1.649(2)	1.785(5)
1	0.373116(1)	0.463(2)	1.653(2)	1.788(3)

TABLE I: Nonuniversal characteristics of the family of SR(S/B)P models.

of models with arbitrary Q based on dynamic maintaining of connectivity on the dual lattice that allowed site or bond deletion operations to be performed in O(1) time. Lattices of size 128-256 turned out to be the most useful, and 10^6–10^7 samples were simulated for each Q. This algorithm enabled us to accurately determine fine differences in the critical exponents, appearing in the problem.

The procedure of finding the indices was as follows (c.f. paper [12], where a similar procedure was proposed for study of backbones for standard percolation): For a given L an ensemble of realizations on $L \times L$ square was generated with an additional "boundary constraint", requiring that the sites and bonds, belonging to two opposite sides of the square (say, upper and lower ones) could not be removed and constitute two "bars" which by convention belong the the backbone. For a given realization of the

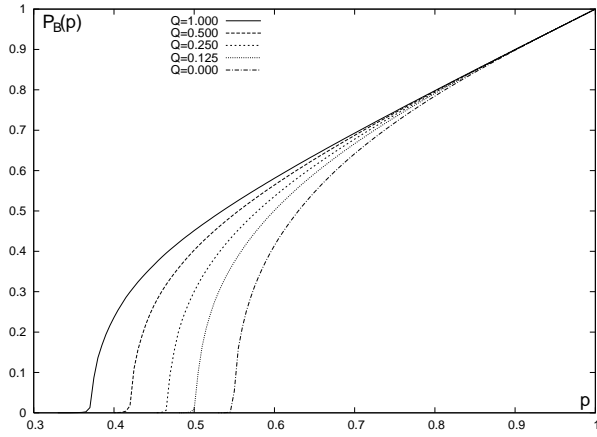| Q	ν_B	$d_{\min}$	$D_{\min}$
0	4/3	1.22(1)	1.13(1)
0.5	1.333(5)	—	—
1	1.337(8)	1.226(12)	1.136(10)

TABLE II: Universal indices of the SR(S/B)P-family.

FIG. 4: The backbone density for SR(S/B)P hybrid models at different values of mixing parameter Q. Concentration p is defined as a number of bonds present in the system; both p and P_B are normalized by the total number of bonds in the full lattice.

FIG. 5: Linear extrapolation of $\beta_{\rm eff}(L)$ to $1/L \to 0$. The rescaling parameter was chosen $a = 3/2$.

process the threshold $\tilde{p}_c$ was defined as a concentration, at which the infinite cluster splits into two parts – one, connected to the upper bar, the other – to the lower bar. A distribution function of $\tilde{p}_c$ over the ensemble was found. The average and the standard deviation of this distribution are (see, e.g., [2]):

$$\overline{p}_c(L) \equiv \overline{\tilde{p}_c} \approx p_c + C_1 L^{-\omega}, \quad \Delta p_c(L) \approx C_2 L^{-1/\nu_B}, \quad (12)$$

and the average value of the backbone density at the transition is

$$\overline{P}_B(L) \equiv \overline{P_B(\tilde{p}_c)} \approx C_3 L^{D_B - D}, \qquad (13)$$

where C_1, C_2, C_3 – critical amplitudes. The values of p_c, ν_B, D_B, β_B, and ω, given in the tables, were extracted from relations (12,13). The extraction procedure is illustrated in Fig.5, where $\beta_{\rm eff}(L) \equiv (\nu/\ln a) \ln[\overline{P}_B(aL)/\overline{P}_B(L)]$, and $a > 1$ is some rescaling parameter (the result is not sensitive to the choice of a). Note, that the exponent ω is *not universal* already within the class of standard percolation (namely, it depends on a particular lattice, as well as on the boundary conditions), so that the dependence of ω on Q may only be viewed as an indirect evidence of non-universality of the SR(S/B)P class.

In conclusion, we have studied a one parametric family of self-repairing site/bond percolation single-cluster models. In all models of this family a topological phase transition between net-like and tree-like phases was found, but the backbone fractal dimension $D_B(Q)$ turned out to be non-universal. It apparently is a smooth monotonous function of the site/bond mixing parameter Q, varying by ~ 0.01 in the range $0 < Q < 1$. Though the above variation is quite small, it is well outside the error bars ± 0.002 of our calculations. To reach this high accuracy special efforts were made.
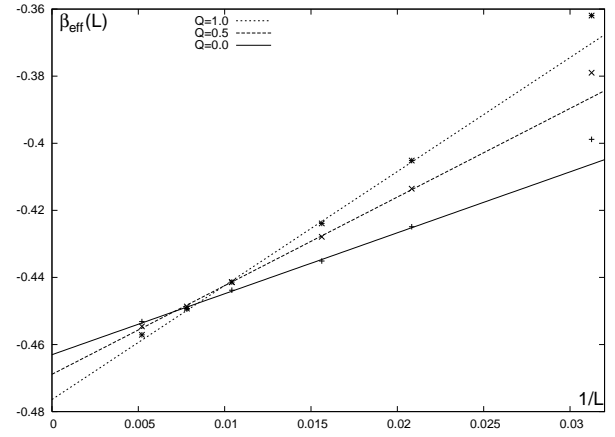
The non-universality of the critical indices was not unexpected, since there are no known reasons for universality, similar to those, existing for the family of standard percolation problems. In this connotation, it was a great surprise that index ν_B of the backbone correlation radius; the graph dimension $D_{\rm min}$ of the backbone; and the graph dimension $d_{\rm min}$ of the tree-like phase seemed to be Q-independent within our (rather high) accuracy. Based on our numerical observations, we conjecture:

$$\nu_B^{\rm (SR(S/B)P)}(Q) \equiv \nu^{\rm (perc)}, \quad D_{\rm min}^{\rm (SR(S/B)P)}(Q,p) \equiv D_{\rm min}^{\rm (perc)},$$
$$d_{\rm min}^{\rm (SR(S/B)P)}(Q,p) \equiv d_{\rm min}^{\rm (MST)}.$$

So far, we did not find any rational explanation for this intriguing phenomenon of "partial universality".

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